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Equivalence of the super Lax and local Dunkl operators for Calogero-like models

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Abstract

Following Shastry and Sutherland I construct the super Lax operators for the Calogero model in the oscillator potential. These operators can be used for the derivation of the eigenfunctions and integrals of motion of the Calogero model and its supersymmetric version. They allow us to infer several relations involving the Lax matrices for this model in a fast way. It is shown that the super Lax operators for the Calogero and Sutherland models can be expressed in terms of the supercharges and so-called local Dunkl operators constructed in our recent paper with M Ioffe. Several important relations involving Lax matrices and Hamiltonians of the Calogero and Sutherland models are easily derived from the properties of Dunkl operators.

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1. Introduction

The most well-known exactly solvable and integrable quantum systems of N particles on a line are given in [1, 2]. One of them is the Calogero model [3–5], with the Hamiltonian

$$H = -\Delta + \omega^2 \sum_{i=1}^{N} x_i^2 + \sum_{i\neq j}^{N} \frac{l^2 - l}{(x_i - x_j)^2}. \tag{1}$$

When $\omega = 0$ this model is called the free Calogero or Calogero–Moser [6] one (following the notations of [7]).

Another is the trigonometric Sutherland or TS model [8–11] with the Hamiltonian

$$H = -\Delta + \sum_{i\neq j}^{N} \frac{l^2 - l}{\sin^2(x_i - x_j)}. \tag{2}$$

There is also a hyperbolic variant of the Sutherland model (HS) [8], where there is a hyperbolic sine in the denominator. For brevity we will call the above three models the Calogero-like ones. These models correspond to the A_{N-1} root system; generalizations for other root systems also exist [1, 12–18].

The formalism of quantum Lax operators [1, 2, 7, 8, 12, 19–22] plays an important role in the proof of the integrability of the Calogero-like models and derivation of their eigenfunctions.

The supersymmetric [23, 24] generalization of the Calogero model was constructed in [18, 25, 26] and that of the Sutherland model was considered in [27, 28].

In the paper [27], the super Lax operators were set forth. These operators are bilinears in the fermionic variables, the coefficients being the standard quantum Lax matrices. The super Lax operators allow one to derive the standard relations involving the Lax matrices in a faster and simpler way.

Apart from the Lax formalism there is another powerful approach to the proof of the integrability and exact solvability of the Calogero-like models that uses the Dunkl operators [29–34]. Its supersymmetric generalization was constructed in [28, 35, 36]. In [37] another relation between the Dunkl operators and supersymmetry was considered. Namely, the so-called local Dunkl operators were constructed that intertwined the matrix Calogero-like Hamiltonians corresponding to some irreducible representations of S_N. For the class of Young diagrams described in [38], some local Dunkl operators were found to coincide with the components of the supercharges (after the separation of the centre-of-mass (CM) part in the latter). This derivation is analogous to the projection method of [28] that works only for the supersymmetric models.

The main result of the present work is that another class of the local Dunkl operators of [37] coincides with the CM-independent part of the components of the super Lax operators of [27].

Thus one has a means to construct the Lax operators for a given system provided it possesses a set of Dunkl operators. This can be useful e.g. for the Calogero-like systems for root systems other than A_{N-1} or for elliptic Calogero models.

The paper is organized as follows. In section 2 we briefly review the formalism of the supersymmetric quantum mechanics (SUSY QM) [23, 24] and its application to the Calogero-like models. The super Lax operators for the free models [27] are constructed. The components of these operators in the one-fermionic sector turn out to coincide with the usual Lax matrices. We also construct the super Lax operator for the Calogero model which we believe is new. It can be used for the construction of the eigenstates of the model and for the proof of its integrability. Some useful identities for the total sums of the Lax matrices [12] are formulated. They are to be proved in the subsequent sections.

In section 3 the bosonic [39] and fermionic Jacobi variables with reference to the Calogero-like models [38, 40] are introduced. The separation of the CM part in the superHamiltonian and supercharges [38] is briefly reviewed. It is shown that in the case of the Calogero model one can obtain the identities for the total sums of Lax matrices given in [19, 20] from the properties of the super Lax operators constructed in section 2.

At the beginning of section 4 the local Dunkl operators [37] are presented. The relations in which they intertwine the matrix Hamiltonians for the Calogero-like models are given.

Then a special kind of the Clebsh–Gordan coefficients for the local Dunkl operator of a free Calogero-like model is constructed with the help of fermionic variables. Thus we give an explicit example of the exactly solvable Dirac-like operator of [37]. The new local Dunkl operator can be viewed as a component of a certain super Lax-like operator, bilinear in fermions. This super Lax-like operator turns out to coincide with the CM-independent part of the usual super Lax operator [27] written in the Jacobi variables. Therefore one can infer the fact that the super Lax operator commutes with the superHamiltonian from the intertwining relations of the local Dunkl operators and matrix Hamiltonians derived in [37]. The CM-dependent part of the super Lax operator is expressed in terms of the supercharge operators, which allows us to prove an identity from [19, 20].

Then we use the same Clebsh–Gordan coefficients for the local Dunkl operators for the Calogero model. The result again has the form of components of certain super Lax-like operators. The latter, instead of commuting with the superHamiltonian, will obey oscillator-like commutation relations with it. As in the free case, the new super Lax-like operators coincide with the CM-independent components of the usual super Lax operators written in the Jacobi variables. This again allows one to infer the oscillator-like commutation relations between the super Lax operators and the Hamiltonian from the intertwining relations of the local Dunkl operators and matrix Hamiltonians. For the Calogero model the CM-dependent part of the super Lax operator is again expressed in terms of the supercharge operators, which allows us to prove an identity from [7].

The possible extension of the results of the paper onto the case of the Calogero-like models corresponding to general root systems is briefly discussed in the last subsection.

2. Supersymmetric Calogero-like models

2.1. Multidimensional SUSY QM [24]

The supersymmetric quantum system for arbitrary number of dimensions N consists [24] of the superHamiltonian $\mathcal{H}$ and the supercharges[1]:

$$Q^- \equiv \sum_{j=1}^{N} \psi_j Q_j^+ \qquad Q^+ = (Q^-)^\dagger = \sum_{j=1}^{N} \psi_j^+ Q_j^- \tag{3}$$

with the algebra

$$(Q^+)^2 = (Q^-)^2 = 0 \qquad \mathcal{H} = \{Q^+, Q^-\} \tag{4}$$

$$[\mathcal{H}, Q^\pm] = 0 \tag{5}$$

where ψ_i, $\psi_i^+ = (\psi_i)^\dagger$ are fermionic operators:

$$\{\psi_i, \psi_j\} = 0 \qquad \left\{\psi_i^+, \psi_j^+\right\} = 0 \qquad \left\{\psi_i, \psi_j^+\right\} = \delta_{ij}. \tag{6}$$

The Hamiltonian and supercharge operators act in the tensor product of the fermionic Fock space with the basis

$$\begin{aligned} &\psi_{i_1}^+ \cdots \psi_{i_M}^+ |0\rangle \equiv |i_1 \cdots i_M\rangle \qquad i_1 < \cdots < i_M \leqslant N \quad M \leqslant N \\ &\psi_i |0\rangle = 0 \qquad i \leqslant N \end{aligned} \tag{7}$$

and some bosonic Fock space where the operators $Q_i^\pm$ act. From this moment onwards we will not mention the bosonic Fock space for brevity.

The superHamiltonians condidered in this text conserve the fermionic number $\mathcal{N} \equiv \sum_{j=1}^N \psi_j^+ \psi_j$. Hence, they have the following block-diagonal form in the basis (7)

$$\mathcal{H} = \mathrm{diag}(H^{(0)}, \mathbf{H}^{(1)}, \ldots, \mathbf{H}^{(N-1)}, H^{(N)}) \tag{8}$$

where the matrix operator $\mathbf{H}^{(M)}$ with dimension[2] $C_N^M \times C_N^M$ is the component of $\mathcal{H}$ in the subspace with fixed fermionic number M. The components with M are equal to 0, and N are thus scalar operators, and are not marked by boldface.

[1] Here and below the indices $i, j, k, \ldots$ range from 1 to N.

[2] The C_N^M here are the binomial coefficients.

Table 1. Functions and constants appearing in the supercharge and superHamiltonian.

Name of model	$V(x)$	$V'(x)$	E_0
TS	$l\cot x$	$-l/\sin^2 x$	$-(N-2)(N-1)Nl^2/3$
HS	$l\coth x$	$-l/\sinh^2 x$	$-(N-2)(N-1)Nl^2/3$
Free Calogero	l/x	$-l/x^2$	0

2.2. *Supersymmetric Calogero-like models [25–38]*

The free supersymmetric Calogero-like models are characterized by the bosonic parts of the supercharges (3) of the form

$$Q_l^{\pm} = \mp\partial_l - \sum_{l\neq k}^{N} V(x_l - x_k) \equiv \mp\partial_l - \sum_{l\neq k}^{N} V_{lk} \tag{9}$$

where V(x) are given in table 1 and $V_{lk} \equiv V(x_l - x_k)$.

With such supercharges the superHamiltonian (4) turns into [38]

$$\mathcal{H} = -\Delta + \sum_{i\neq l}^{N} V_{il}^2 + \sum_{i\neq j}^{N} \mathcal{K}_{ij} V'_{ij} - E_0. \tag{10}$$

The constants E_0 are given in table 1. The operator $\mathcal{K}_{ij}$ [27] has the form

$$\begin{aligned}\mathcal{K}_{ij} &\equiv \psi_i^+\psi_j + \psi_j^+\psi_i - \psi_i^+\psi_i - \psi_j^+\psi_j + 1 = 1 - \left(\psi_i^+ - \psi_j^+\right)(\psi_i - \psi_j)\\ &= \mathcal{K}_{ji} = (\mathcal{K}_{ij})^\dagger\end{aligned} \tag{11}$$

and is the fermionic exchange operator:

$$\mathcal{K}_{ij}\psi_i^+ = \psi_j^+\mathcal{K}_{ij} \qquad \mathcal{K}_{ij}\psi_i = \psi_j\mathcal{K}_{ij} \tag{12}$$

$$\mathcal{K}_{ij}\psi_k^+ = \psi_k^+\mathcal{K}_{ij} \qquad \mathcal{K}_{ij}\psi_k = \psi_k\mathcal{K}_{ij} \qquad k \neq i, j. \tag{13}$$

The Calogero model is characterized by the bosonic parts of the supercharges (3) of the form

$$Q_l^{\pm} = \mp\partial_l + \omega x_l - l\sum_{l\neq k}^{N}(x_l - x_k)^{-1}.$$

Accordingly, the superHamiltonian (4) of the model has the form

$$\mathcal{H} = -\Delta + \omega^2\sum_i x_i^2 + \sum_{i\neq j}\frac{l^2 - l\mathcal{K}_{ij}}{(x_i - x_j)^2} + 2\omega\mathcal{N} - \omega(1 + (N-1)(Nl+1)). \tag{14}$$

The exchange operator $\mathcal{K}_{ij}$ in (10) and (14) commutes with $\mathcal{N}$, and therefore assumes a block-diagonal form in the basis (7), similarly to the superHamiltonian:

$$\mathcal{K}_{ij} = \mathrm{diag}\left(T_{ij}^{(0)}, \mathbf{T}_{ij}^{(1)}, \ldots, \mathbf{T}_{ij}^{(N-1)}, T_{ij}^{(N)}\right). \tag{15}$$

The components (8) of the superHamiltonian have the form

$$\mathbf{H}^{(M)} = \left[-\Delta + \sum_{i\neq l}^{N} V_{il}^2 - E_0\right]\mathbf{I} + \sum_{i\neq j}^{N}\mathbf{T}_{ij}^{(M)} V'_{ij}$$

for the free Calogero-like models, and

$$\mathbf{H}^{(M)} = \left[-\Delta + \omega^2 \sum_i^N x_i^2 + \omega(2M - 1 - (N-1)(Nl+1)) \right] \mathbf{I} + \sum_{i \neq j}^N \frac{l^2 \mathbf{I} - l \mathbf{T}_{ij}^{(M)}}{(x_i - x_j)^2} \tag{16}$$

for the Calogero model.

One can easily see that $T_{ij}^{(0)} = 1 = -T_{ij}^{(N)}$. Thus the component $H^{(0)}$ coincides up to an additive constant with the scalar Hamiltonian (2) for the Sutherland model and with (1) for the Calogero model.

The elements of the matrix $\mathbf{T}_{ij}^{(1)}$ have the form

$$\left(T_{ij}^{(1)}\right)_{lk} \equiv \delta_{lk} - \delta_{li}\delta_{ki} - \delta_{lj}\delta_{kj} + \delta_{li}\delta_{kj} + \delta_{lj}\delta_{ki}. \tag{17}$$

2.3. The super Lax operators

As noted in [27], the superHamiltonian (10) satisfies the following commutation relation:

$$[\mathcal{H}, \mathcal{L}] = 0 \tag{18}$$

where the operator $\mathcal{L}$ is the so-called super Lax operator given by

$$\mathcal{L} = L_{km}\psi_k^+\psi_m \qquad L_{km} = -\mathrm{i}\partial_k\delta_{km} + \mathrm{i}(1 - \delta_{km})V_{km}. \tag{19}$$

Here, L_{km} are the elements of the well-known Lax matrix $\mathbf{L}$, and ψ_k^+, ψ_m are the fermionic operators (6).

In section 4.2 of this paper we present an alternative proof of (18) using the Dunkl operators.

One may also note that $[\mathcal{N}, \mathcal{L}] = 0$, so the super Lax operator conserves the fermionic number and has the block-diagonal form

$$\mathcal{L} = \mathrm{diag}(0, \mathbf{L}^{(1)}, \ldots, \mathbf{L}^{(N-1)}, L^{(N)}). \tag{20}$$

Note that $L^{(N)} = -\mathrm{i}\sum_k \partial_k$.

We will use below the following consequence of the anticommutaion relations (6). For a fermionic quantity

$$\mathcal{A} = \sum_{k,l} A_{kl}\psi_k^+\psi_l \tag{21}$$

the matrix elements in the one-fermionic sector are

$$\langle i|\mathcal{A}|j\rangle = A_{kl}\langle 0|\psi_i\psi_k^+\psi_l\psi_j^+|0\rangle = A_{ij} \tag{22}$$

so its first block on the diagonal in the form (20) is $\mathbf{A}^{(1)} = \mathbf{A}$. For example, $\mathbf{L}^{(1)} = \mathbf{L}$.

The standard relation involving the Lax matrices is

$$[\mathbf{L}, H^{(0)}] = [\mathbf{M}, \mathbf{L}] \tag{23}$$

where $H^{(0)}$ is the Hamiltonian of a scalar free Calogero-like model, and the elements of $\mathbf{M}$ have the form

$$M_{lk} = 2(1 - \delta_{lk})V'_{lk} - 2\delta_{lk}\sum_{j \neq k} V'_{kj}. \tag{24}$$

Equation (23) was shown in [27] to follow from (18), but not vice versa.

The Lax matrix for the free Calogero-like models satisfies the following identity [21]:

$$\mathrm{Ts}(\mathbf{L}^2) = H^{(0)} \tag{25}$$

which is used in the proof of integrability of the free Calogero-like models [22]. For a matrix $\mathbf{A}$ the total sum Ts is defined as

$$\mathrm{Ts}\mathbf{A} = \sum_{i,j=1}^{N} A_{ij}.$$

Equation (25) will also be proved in section 4.2.

The following identity is also true [27]:

$$[H^{(0)}, I_n] = 0 \qquad I_n = \mathrm{Ts}\mathbf{L}^n. \tag{26}$$

The involution of the quantities I_n is proved in [19, 20].

It turns out that the construction of the super Lax operators is possible for the Calogero model too. To the author's knowledge this construction has not been proposed before; thus the rest of the subsection contains new material. Namely, define the following fermionic operator:

$$\mathcal{L}^{\pm} = L^{\pm}_{km}\psi^{+}_{k}\psi_m \qquad L^{\pm}_{km} = L_{km} \pm \mathrm{i}\omega x_k\delta_{km} \qquad L_{km} = -\mathrm{i}\partial_k\delta_{km} + \mathrm{i}\frac{1-\delta_{km}}{x_k - x_m} \tag{27}$$

where L_{km} are the elements of the Lax matrix for the free Calogero model. The operators (27) and the superHamiltonian (14) satisfy the following generalization of (18):

$$[\mathcal{H}, \mathcal{L}^{\pm}] = \pm 2\omega\mathcal{L}^{\pm}. \tag{28}$$

The proof of these relations can be found in subsection 4.3. Equation (28) describes an oscillator-like algebra and hence can be used for the construction of the spectrum of the superHamiltonian (14) and proof of its integrability. Namely, the ground-state wavefunction for the (super)Calogero Hamiltonian (1), (14) is [3, 25]:

$$\psi_0 = \exp\left(-\frac{\omega}{2}\sum_{j=1}^{N} x_j^2\right)\prod_{i<k}^{N}|x_i - x_j|^{l}. \tag{29}$$

Applying powers of the operators (3) and (27) to this wavefunction, one can get the excited states of $\mathcal{H}$. The integrals of $\mathcal{H}$ are linear combinations of the monomials in $Q^{\pm}$, $\mathcal{L}^{\pm}$ in which the power of $\mathcal{L}^{+}$ is equal to the power of $\mathcal{L}^{-}$. Examples of such are

$$\mathcal{L}_1 = \mathcal{L}^{+}\mathcal{L}^{-} \qquad \mathcal{L}_2 = \mathcal{L}^{-}\mathcal{L}^{+}. \tag{30}$$

Similarly to the free Calogero models, $[\mathcal{N}, \mathcal{L}^{\pm}] = 0$, so

$$\mathcal{L}^{\pm} = \mathrm{diag}(0, \mathbf{L}^{(1)\pm}, \ldots, \mathbf{L}^{(N-1)\pm}, L^{(N)\pm}).$$

It follows from (22) that $\mathbf{L}^{(1)\pm} = \mathbf{L}^{\pm}$, where $\mathbf{L}^{\pm}$ is the matrix with the elements (27).

One can infer the usual relations [21] involving the Lax matrices from (28) in the following way:

$$\pm 2\omega\mathbf{L}^{\pm} = [\mathbf{L}^{\pm}, \mathbf{H}^{(1)}] = [\mathbf{L}^{\pm}, \mathbf{H}^{(1)} - \mathbf{I}H^{(0)} + \mathbf{I}H^{(0)}]$$

where $\mathbf{H}^{(1)}$ is the first component (16) of the superHamiltonian. Hence,

$$[H^{(0)}, \mathbf{L}^{\pm}] = [\mathbf{L}^{\pm}, \mathbf{M}] \pm 2\omega\mathbf{L}^{\pm}$$

where $\mathbf{M} \equiv \mathbf{H}^{(1)} - \mathbf{I}(H^{(0)} + 2\omega)$ is the same standard matrix (24) as in the free case. It follows from (16) and (17) that its elements are

$$M_{mk} = 2l(\delta_{mk} - 1)(x_m - x_k)^{-2} + 2l\delta_{mk}\sum_{j\neq k}(x_k - x_j)^{-1}.$$

For the Calogero model we can derive an analogue of (25). Namely, define the quantities

$$\mathbf{L}_1 = \mathbf{L}^{+}\mathbf{L}^{-} \qquad \mathbf{L}_2 = \mathbf{L}^{-}\mathbf{L}^{+}.$$

The matrices $\mathbf{L}_j$ are the components of the operators $\mathcal{L}_j$ (30) in the sector $\mathcal{N} = 1$. It turns out [21] that

$$H^{(0)} = \mathrm{Ts}\mathbf{L}_1 = \mathrm{Ts}\mathbf{L}_2 + \text{const.} \tag{31}$$

A variant of the proof can be found in subsection 4.3.

An analogue of (26) can also be proved:

$$[H^{(0)}, I_{jn}] = 0 \qquad I_{jn} = \mathrm{Ts}\mathbf{L}_j^n \qquad j = 1, 2; \tag{32}$$

a proof is given in section 3. The involution of the quantities I_{jn} is proved in [7, 12].

One can also define the following operators [21]

$$O_p^m = \mathrm{Ts}((\mathbf{L}^-)^m(\mathbf{L}^+)^p) \tag{33}$$

that commute with $H^{(0)}$ as

$$\left[H^{(0)}, O_p^m\right] = 2(p-m)\omega O_p^m. \tag{34}$$

The proof is again given in section 3. Applying the operators (33) to the ground-state wavefunction (29) one gets the excited states of $H^{(0)}$ for the Calogero model.

3. The Jacobi variables and SUSY QM

3.1. Definitions [38]

The bosonic [39] and fermionic [38] Jacobi variables are defined as

$$y_k = R_{km}x_m \qquad \phi_k = R_{kl}\psi_l \tag{35}$$

where R_{kl} is a real orthogonal matrix; see [38] for details. In this text it will be important for us that $R_{Nl} = N^{-1/2}$, i.e.,

$$y_N = \frac{1}{\sqrt{N}}\sum_{i=1}^{N} x_i \qquad \phi_N = \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\psi_i.$$

The new fermionic variables (35) satisfy the standard anticommutation relations:

$$\{\phi_k, \phi_l\} = 0 \qquad \left\{\phi_k^+, \phi_l^+\right\} = 0 \qquad \left\{\phi_k, \phi_l^+\right\} = \delta_{kl}. \tag{36}$$

With the help of the fermionic Jacobi variables one can separate the centre-of-mass term in the supercharges (3) in the following way [38]:

$$Q^{\pm} = q^{\pm} + Q_C^{\pm} \qquad \mathcal{H} = h + H_C \tag{37}$$

where

$$Q_C^- \equiv -\phi_N\frac{\partial}{\partial y_N} \qquad Q_C^+ = \phi_N^+\frac{\partial}{\partial y_N} \qquad H_C = -\partial^2\big/\partial y_N^2 \tag{38}$$

for the free models, and

$$Q_C^- \equiv \phi_N Q_N^+ \qquad Q_C^+ \equiv \phi_N^+ Q_N^- \qquad Q_N^{\pm} = \mp\frac{\partial}{\partial y_N} + \omega y_N \tag{39}$$

$$H_C = -\mathrm{d}^2\big/\mathrm{d}y_N^2 + \omega^2 y_N^2 + \omega\big(2\phi_N^+\phi_N - 1\big) \tag{40}$$

for the Calogero model.

These new quantities satisfy the relations of the following superalgebra [38]:

$$\begin{aligned}
&(q^{\pm})^2 = \left(Q_C^{\pm}\right)^2 = \left\{q^{\pm}, Q_C^{\pm}\right\} = 0\\
&\{q^+, q^-\} = h \qquad \left\{Q_C^+, Q_C^-\right\} = H_C \qquad [h, H_C] = 0\\
&[\mathcal{H}, q^{\pm}] = [h, q^{\pm}] = [H_C, q^{\pm}] = [\mathcal{H}, {Q_C}^{\pm}] = [h, {Q_C}^{\pm}] = [H_C, {Q_C}^{\pm}] = 0.
\end{aligned} \tag{41}$$

3.2. Application to the Lax operators

If one uses the fermionic Jacobi variables, it is natural to go from the basis (7) to a new one[3] (here we follow [38]):

$$\begin{aligned} &\phi^+_{\beta_1}\cdots\phi^+_{\beta_M}|0\rangle \equiv |\beta_1\cdots\beta_M\rangle \equiv |\beta\rangle \qquad \phi^+_N\phi^+_{\beta_1}\cdots\phi^+_{\beta_M}|0\rangle \equiv |N\beta\rangle \\ &\beta_1 < \cdots < \beta_M \qquad M < N. \end{aligned} \tag{42}$$

In the new basis (42), the superHamiltonian (10) of the free Calogero-like models will take the form

$$\mathcal{H} = \mathrm{diag}(\widetilde{H}^{(0)}, \widetilde{\mathbf{H}}^{(1)}, \ldots, \widetilde{\mathbf{H}}^{(N-2)}, \widetilde{H}^{(N-1)}, \widetilde{H}^{(0)}, \widetilde{\mathbf{H}}^{(1)}, \ldots, \widetilde{\mathbf{H}}^{(N-2)}, \widetilde{H}^{(N-1)})$$

where

$$\widetilde{\mathbf{H}}^{(M)} = \left[-\Delta + \sum_{i\neq l}^{N} V_{il}^2 - E_0\right]\mathbf{I} + \sum_{i\neq j}^{N} \widetilde{\mathbf{T}}^{(M)}_{ij} V'_{ij} \tag{43}$$

and $\widetilde{\mathbf{T}}^{(M)}_{ij}$ are matrices[4] with the elements

$$\left(\widetilde{T}^{(M)}_{ij}\right)_{\gamma\beta} = \langle\gamma_M\cdots\gamma_1|\mathcal{K}_{ij}|\beta_1\cdots\beta_M\rangle \tag{44}$$

where $\mathcal{K}_{ij}$ is the fermionic exchange operator (11). It is proved in [38] that such matrices form the representation[5] of S_N with the Young diagram $(N-M, 1^M)$.

The superHamiltonian (14) of the Calogero model in the Jacobi basis will have the form

$$\begin{aligned} \mathcal{H} = \mathrm{diag}(&\widetilde{H}^{(0)}, \widetilde{\mathbf{H}}^{(1)}, \ldots, \widetilde{\mathbf{H}}^{(N-2)}, \widetilde{H}^{(N-1)}, \\ &\widetilde{H}^{(0)} + 2\omega, \widetilde{\mathbf{H}}^{(1)} + 2\mathbf{I}\omega, \ldots, \widetilde{\mathbf{H}}^{(N-2)} + 2\mathbf{I}\omega, \widetilde{H}^{(N-1)} + 2\omega) \end{aligned} \tag{45}$$

where

$$\widetilde{\mathbf{H}}^{(M)} = \left[-\Delta + \omega^2\sum_{i}^{N} x_i^2 + \omega(2M - 1 - (N-1)(Nl+1))\right]\mathbf{I} + \sum_{i\neq j}^{N} \frac{l^2\mathbf{I} - l\widetilde{\mathbf{T}}^{(M)}_{ij}}{(x_i - x_j)^2}.$$

Now one can use the above formalism from [38] to prove (34) in the same way as (26) was proved in [27].

First take into account that for any fermionic quantity $\mathcal{A}$ that commutes with $\mathcal{N}$,

$$\mathrm{Ts}\mathbf{A}^{(1)} = \sum_{i,j=1}^{N} A^{(1)}_{ij} = \sum_{i,j}\langle i|\mathcal{A}|j\rangle = N\langle N|\mathcal{A}|N\rangle \tag{46}$$

where $\mathbf{A}^{(1)}$ is the component of $\mathcal{A}$ in the sector with $\mathcal{N} = 1$. Then, define the quantities

$$\mathcal{O}^m_p = (\mathcal{L}^-)^m(\mathcal{L}^+)^p.$$

It follows from (28) that

$$\left[\mathcal{H}, \mathcal{O}^m_p\right] = 2(p-m)\omega\mathcal{O}^m_p.$$

From (46) we get

$$O^m_p = \mathrm{Ts}[(\mathbf{L}^-)^m(\mathbf{L}^+)^p] = N\langle N|\mathcal{O}^m_p|N\rangle.$$

[3] The indices of the Jacobi variables denoted by Greek letters range from 1 to $N-1$; those denoted by Latin letters range from 1 to N.

[4] One should not confuse $\widetilde{\mathbf{T}}^{(M)}_{ij}$ with $\mathbf{T}^{(M)}_{ij}$ from (15) which corresponds to a *reducible* representation of S_N.

[5] We will denote the irreducible representations of S_N by their Young diagrams. The standard notation [41] for the Young diagram containing λ_i cells in the ith line is $(\lambda_1, \ldots, \lambda_n)$; if a diagram contains m identical lines with μ cells, it is denoted by $(\ldots, \mu^m, \ldots)$.

Therefore,

$$
\begin{aligned}
2(p-m)\omega O_p^m &= 2(p-m)\omega N\langle N|\mathcal{O}_p^m|N\rangle = N\langle N|\big[\mathcal{H}, \mathcal{O}_p^m\big]|N\rangle = N\langle N|\mathcal{H}\mathcal{O}_p^m|N\rangle \\
&\quad - N\langle N|\mathcal{O}_p^m\mathcal{H}|N\rangle = (H^{(0)}+2\omega)N\langle N|\mathcal{O}_p^m|N\rangle - N\langle N|\mathcal{O}_p^m|N\rangle(H^{(0)}+2\omega) \\
&= \big[H^{(0)}, O_p^m\big].
\end{aligned}
$$

In exactly the same way one can deduce (32) from (18).

4. Connection between the local Dunkl operators and the super Lax operators

4.1. Intertwining relations involving the local Dunkl operators

In a recent paper [37] we considered the matrix Calogero-like Hamiltonians of the form

$$
\mathbf{H}^A = \left[-\Delta + \sum_{i\neq l}^{N} V_{il}^2\right]\mathbf{I} + \sum_{i\neq j}^{N} \mathbf{T}_{ij}^A V_{ij}' \tag{47}
$$

where A is an irreducible representation of the group S_N of permutations of N particles, and $\mathbf{T}_{ij}^A$ are the matrices of this representation.

We will need below the representation L with the Young diagram $(N-1, 1)$. Suppose we have an irreducible representation A of S_N, such that the interior product $L \times A$ contains A. Then the following commutation relation is true [37]:

$$
[\mathbf{H}^A, \mathbf{D}^{AA}] = 0 \tag{48}
$$

where $\mathbf{D}^{AA}$ is the so-called local Dunkl operator. It is a $\dim A \times \dim A$ matrix with elements

$$
D_{\sigma\alpha}^{AA} = (\xi\beta|\sigma)R_{\xi k}\left[-\mathrm{i}\partial_k\delta_{\beta\alpha} + \mathrm{i}\sum_{m\neq k} V_{km}\big(T_{km}^A\big)_{\beta\alpha}\right]. \tag{49}
$$

Here, $R_{\xi k}$ is the matrix of transition from the particle coordinates to the Jacobi ones; $(\xi\beta|\sigma) \equiv (L\xi, A\beta|A\sigma)$ are the Clebsh–Gordan coefficients for the contribution of A in $L \times A$.

Note that the SUSY QM intertwining relations for the Calogero-like systems can also be deduced from the local Dunkl operators [37].

It was proved in [38] that for the TS model, equation (48) allows us to find the spectrum of $\mathbf{D}^{AA} \equiv \mathbf{D}^A$ because we know the spectrum of $\mathbf{H}^A$. However, definition (49) of $\mathbf{D}^A$ contains a Clebsh–Gordan coefficient $(L\xi, A\beta|A\sigma)$ that is relatively hard to find, except for the cases discussed below and in [37].

In the case of the Calogero model, the following analogue of (48) was set forth in [37]:

$$
[\mathbf{H}^A, \mathbf{D}^{AA\pm}] = \pm 2\omega\mathbf{D}^{AA\pm} \tag{50}
$$

where $\mathbf{H}^A$ is the matrix Calogero Hamiltonian for the representation A:

$$
\mathbf{H}^A = \left[-\Delta + \omega^2\sum_i x_i^2 + \sum_{i\neq j}\frac{l^2}{(x_i-x_j)^2} + N\omega\right]\mathbf{I} - \sum_{i\neq j}\left(\frac{l}{(x_i-x_j)^2} + a\right)\mathbf{T}_{ij}^A \tag{51}
$$

and the elements of the matrix $\mathbf{D}^{AA\pm}$ are

$$
D_{\sigma\alpha}^{AA\pm} = (\xi\beta|\sigma)R_{\xi j}\left[(-\mathrm{i}\partial_j \pm \mathrm{i}\omega x_j)\delta_{\beta\alpha} + \mathrm{i}l\sum_{m\neq j}\frac{\big(T_{jm}^A\big)_{\beta\alpha}}{x_j - x_m}\right]. \tag{52}
$$

We will see in section 4.3 that the components of the operators $\mathcal{L}^\pm$ can be reduced to a partial case of (52) (see equations (69) and (70)).

4.2. The connection between the local Dunkl operators and the super Lax ones

In this subsection we are to prove that the super Lax operator (19) can be expressed in terms of the local Dunkl operators (49), and the commutation relations (18) follow from (48).

Let us suppose that

$$A = (N - M, 1^M) \tag{53}$$

in (48) and (49). Then we can define the Clebsh–Gordan coefficients in (49) in the following way:

$$(1\xi, M\beta|M\zeta) = \langle\zeta|C_\xi|\beta\rangle \qquad C_\xi \equiv R_{\xi k}\psi_k^+\psi_k = C_\xi^\dagger \tag{54}$$

where ψ_i, ψ_i^+ are the fermionic variables satisfying (6); $R_{\xi k}$ is the matrix of transition (35), from the particle coordinates to the Jacobi ones; $|\beta\rangle$ are the states from (the first half of) the basis (42), such that $\mathcal{N}|\beta\rangle = M|\beta\rangle$; $\phi_N|\beta\rangle = 0$.

This is possible because the coefficients (54) satisfy the following characteristic condition of the Clebsh–Gordan coefficients [41]:

$$\big(T_{ij}^L\big)_{\alpha\xi}\big(\widetilde{T}_{ij}^{(M)}\big)_{\gamma\beta}(1\xi, M\beta|M\zeta) = (1\alpha, M\gamma|M\nu)\big(\widetilde{T}_{ij}^{(M)}\big)_{\nu\zeta} \tag{55}$$

where $\widetilde{\mathbf{T}}_{ij}^{(M)}$ is the matrix (44), and

$$\mathbf{T}_{ij}^L = \widetilde{\mathbf{T}}_{ij}^{(1)} \tag{56}$$

is a matrix from the representation L. The proof of (55) can be found in appendix B.

As shown in [38], for the representations from the class (53) one can go from the matrix Hamiltonian $\mathbf{H}^A$ (47) in (49) to $\widetilde{\mathbf{H}}^{(M)} = \mathbf{H}^A - E_0\mathbf{I}$ that is given by (43), E_0 being given in table 1.

Now we can plug the Clebsh–Gordan coefficients (54) into definition (49). After some algebra (see appendix C for details) we arrive at the equality

$$\begin{aligned} D_{\sigma\alpha}^A \equiv D_{\sigma\alpha}^{(M)} &= \langle\sigma|C_\xi|\beta\rangle R_{\xi k}\left[-\mathrm{i}\partial_k\delta_{\beta\alpha} + \mathrm{i}\sum_{m\neq k} V_{km}\big(\widetilde{T}_{km}^{(M)}\big)_{\beta\alpha}\right] \\ &= \langle\sigma|\mathcal{L}|\alpha\rangle + \mathrm{i}N^{-1/2}M\frac{\partial}{\partial y_N}\delta_{\sigma\alpha} \end{aligned} \tag{57}$$

where $\mathcal{L}$ is the super Lax operator (19). Thus we see that the matrix elements of the local Dunkl operator in the basis (42) coincide with the matrix elements of the super Lax operator up to a scalar term.

The operator (57) has the structure of a matrix element connecting two fermionic basis states $\langle\alpha|$ and $|\beta\rangle$. It is natural to consider a fernionic operator built from these matrix elements:

$$\mathcal{D} = \sum_{M,\sigma,\alpha} D_{\sigma\alpha}^{(M)}\big[|\sigma\rangle\langle\alpha| + |N\sigma\rangle\langle\alpha N|\big]. \tag{58}$$

In (58) and all formulae below, the states $|\sigma\rangle$, $|\alpha\rangle$ have fermionic number M, if not specified otherwise.

It follows from (48) that the operator (58) commutes with the superHamiltonian (10).

The components $\mathbf{D}^{(M)}$ (57) of $\mathcal{D}$ have smaller dimension than $\mathbf{L}^{(M)}$, i.e., the block-diagonal structure of $\mathcal{D}$ is more detailed than that of $\mathcal{L}$. Note that $D^{(0)} = 0$.

After a couple of pages of calculations we can conclude that

$$\mathcal{D} = \mathcal{L} + \mathrm{i}N^{-1/2}\left[Q^+\phi_N - \phi_N^+Q^- + \big(\mathcal{N} - 2\phi_N^+\phi_N\big)\frac{\partial}{\partial y_N}\right]. \tag{59}$$

The details are given in appendix D.

Equation (59) gives a simple form of the operator $\mathcal{D}$, and its components $\mathbf{D}^{(M)}$ that can be considered as exactly solvable Dirac-like operators.

It immediately follows from (59) that $[\mathcal{L}, \mathcal{H}] = 0$ since all other operators in (59) have already been seen to commute with $\mathcal{H}$. The only nontrivial commutaion relation of this kind: $[\phi_N, \mathcal{H}] = \left[\phi_N^+, \mathcal{H}\right] = 0$ follows from (37) and (38).

One can also check that

$$[h, \mathcal{D}] = 0 \tag{60}$$

where $h = \mathcal{H} + \frac{\partial^2}{\partial y_N^2}$ is the centre-of-mass independent part (37) of the superHamiltonian.

Equation (60) means that $\mathcal{D}$ plays the same role for h as $\mathcal{L}$ does for $\mathcal{H}$. However, h and $\mathcal{D}$ do not depend on the CM variables[6] y_N, ϕ_N and ϕ_N^+. Thus we have obtained a separation of variables in the (super) Lax operators.

One can also go from the Dunkl operators to the Lax ones by using the approach [28] that does not employ Jacobi variables. However, then it would be difficult to get separation of the centre-of-mass coordinate, and obtain the operators $\mathcal{D}$.

The centre-of-mass terms can also be separated in the supercharges in (59), according to (37). The result will be

$$\mathcal{L} = \mathcal{D} + \mathrm{i}N^{-1/2}\left[\phi_N^+ q^- - q^+\phi_N - \mathcal{N}\frac{\partial}{\partial y_N}\right]. \tag{61}$$

Equation (61) can be used for the derivation of (25):

$$\mathrm{Ts}(\mathbf{L}^2) = H^{(0)}.$$

Namely, taking into account (46), we get

$$\begin{aligned}
\mathrm{Ts}(\mathbf{L}^2) &= N\langle N|\mathcal{L}^2|N\rangle = N\langle N|\left[\mathcal{D} + \mathrm{i}N^{-1/2}\left(\phi_N^+ q^- - q^+\phi_N - \mathcal{N}\frac{\partial}{\partial y_N}\right)\right]^2|N\rangle \\
&= N\langle N|\mathrm{i}N^{-1/2}\left[\phi_N^+ q^- - \mathcal{N}\frac{\partial}{\partial y_N}\right]\mathrm{i}N^{-1/2}\left[-q^+\phi_N - \mathcal{N}\frac{\partial}{\partial y_N}\right]|N\rangle \\
&= -\langle N| - \phi_N^+ q^- q^+ \phi_N + \mathcal{N}^2\frac{\partial^2}{\partial y_N^2}|N\rangle = \langle 0|q^- q^+ + q^+ q^- - \frac{\partial^2}{\partial y_N^2}|0\rangle \\
&= \langle 0|\mathcal{H}|0\rangle = H^{(0)}
\end{aligned}$$

where we have used the fact that $\mathcal{D}|N\rangle = 0$; $\langle N|\mathcal{D} = 0$, because $D^{(0)} = 0$.

We see that (25) actually follows from the supersymmetry of the model.

4.3. The super Lax operators for the Calogero model

It will be convenient below to rewrite the super Lax operator (27) of the Calogero model in the form

$$\mathcal{L}^{\pm} = \mathcal{L} \pm \delta\mathcal{L} \qquad \delta\mathcal{L} \equiv \mathrm{i}\omega x_k \psi_k^+ \psi_k$$

where $\mathcal{L}$ is the super Lax operator (19) for the Calogero model without the harmonic term.

Similarly one can rewrite the local Dunkl operator (52) as

$$\mathbf{D}^{AA\pm} \equiv \mathbf{D}^{AA} \pm \delta\mathbf{D}^{AA}$$

where $\mathbf{D}^{AA}$ is the local Dunkl operator (49) for the free Calogero model, and $\delta\mathbf{D}^{AA}$ is the operator with the elements

$$\delta D^{AA}_{\sigma\alpha} = (L\xi, A\alpha|A\sigma)R_{\xi k}\mathrm{i}\omega x_k.$$

[6] In the case of $\mathcal{D}$ it can be proved by rewriting the operator (19) in the Yacobi variables and using (61).

For the case $A = (N - M, 1^M)$ and the choice (54) of the Clebsh–Gordan coefficients we have equation (57). Similar relation is true for $\delta\mathbf{D}^{AA} \equiv \delta\mathbf{D}^{(M)}$ and $\delta\mathcal{L}$:

$$\delta D^{(M)}_{\gamma\delta} = \langle\gamma|\delta\mathcal{L}|\delta\rangle - \mathrm{i}\omega N^{-1/2} M y_N \delta_{\sigma\alpha}.$$

The proof is completely similar to that of (57), so we omit it.

Similarly to the free case, one can define the operators

$$\begin{aligned}\delta\mathcal{D} &\equiv \sum_{M,\sigma,\alpha} \delta D^{(M)}_{\sigma\alpha}\big[|\sigma\rangle\langle\alpha| + |N\sigma\rangle\langle\alpha N|\big]\\ \mathcal{D}^{\pm} &\equiv \mathcal{D} \pm \delta\mathcal{D} = \sum_{M,\sigma,\alpha} D^{(M)\pm}_{\sigma\alpha}\big[|\sigma\rangle\langle\alpha| + |N\sigma\rangle\langle\alpha N|\big]\end{aligned} \tag{62}$$

where $\mathbf{D}^{(M)}$ is the local Dunkl operator for the free Calogero model, and

$$\mathbf{D}^{(M)\pm} \equiv \mathbf{D}^{(M)} \pm \delta\mathbf{D}^{(M)}.$$

Then it follows from (50) that

$$[\mathcal{H}, \mathcal{D}^{\pm}] = \pm 2\omega\mathcal{D}^{\pm} \tag{63}$$

if we go from the Hamiltonian (51) to (45), following [38], as in the free case.

The calculation of $\delta\mathcal{D}$ is completely similar to that of $\mathcal{D}$ in appendix D; mainly, it amounts to using (D.6) again. Thus we present here only the result:

$$\delta\mathcal{D} = \delta\mathcal{L} - \mathrm{i}N^{-1/2}\big[\delta Q^{+}\phi_N + \phi_N^{+}\delta Q^{-}\big] + \mathrm{i}\omega N^{-1/2} y_N\big[2\phi_N^{+}\phi_N - \mathcal{N}\big] \tag{64}$$

where

$$\delta Q^{-} \equiv w\sum_k x_k\psi_k \qquad \delta Q^{+} \equiv w\sum_k x_k\psi_k^{+}.$$

Plugging (58) and (64) into (62),we get

$$\mathcal{D}^{\pm} = \mathcal{L}^{\pm} + \mathrm{i}N^{-1/2}\left[\big(Q_f^{+} \mp \delta Q^{+}\big)\phi_N - \phi_N^{+}\big(Q_f^{-} \pm \delta Q^{-}\big) + \big(\mathcal{N} - 2\phi_N^{+}\phi_N\big)\left(\frac{\partial}{\partial y_N} \mp \omega y_N\right)\right] \tag{65}$$

where we mark the supercharges of the free Calogero model by the letter f. One can show that the supercharges of the Calogero model in the oscillatory potential can be written as

$$Q^{\pm} = Q_f^{\pm} + \delta Q^{\pm} \qquad \hat{Q}^{\pm} = Q_f^{\pm} - \delta Q^{\pm}$$

where $\hat{Q}^{\pm}$ are the supercharges with different sign of ω.

Then it follows from (65) that

$$\mathcal{D}^{+} = \mathcal{L}^{+} + \mathrm{i}N^{-1/2}\big[\hat{Q}^{+}\phi_N - \phi_N^{+}Q^{-} - \big(\mathcal{N} - 2\phi_N^{+}\phi_N\big)Q_N^{+}\big] \tag{66}$$

$$\mathcal{D}^{-} = \mathcal{L}^{-} + \mathrm{i}N^{-1/2}\big[Q^{+}\phi_N - \phi_N^{+}\hat{Q}^{-} + \big(\mathcal{N} - 2\phi_N^{+}\phi_N\big)Q_N^{-}\big]. \tag{67}$$

As in the free case, it is helpful to separate the centre-of-mass in the supercharges according to (37). In addition to (37) and (39), one will then have for the quantities with inverted sign of ω,

$$\hat{Q}^{\pm} = \hat{q}^{\pm} + \hat{Q}_C^{\pm} \qquad \hat{Q}_C^{-} = -\phi_N Q_N^{-} \qquad \hat{Q}_C^{+} = -\phi_N^{+} Q_N^{+}. \tag{68}$$

Plugging (37), (39) and (68) into (66) and (67), we get

$$\mathcal{L}^{+} = \mathcal{D}^{+} + \mathrm{i}N^{-1/2}\big[\phi_N^{+}q^{-} - \hat{q}^{+}\phi_N - \mathcal{N}Q_N^{+}\big] \tag{69}$$

$$\mathcal{L}^{-} = \mathcal{D}^{-} + \mathrm{i}N^{-1/2}\big[\phi_N^{+}\hat{q}^{-} - q^{+}\phi_N + \mathcal{N}Q_N^{-}\big]. \tag{70}$$

Now we are finally able to prove (28) using (69), (70) and (63). We will consider only the commutation with $\mathcal{L}^{+}$ because the other one is just its Hermitean conjugation.

We will show that all the terms in the operator (69) commute with the superHamiltonian in accordance with (28). The first nontrivial commutator of that kind is

$$\left[\mathcal{H}, \phi_N^+ q^-\right] = \left[\mathcal{H}, \phi_N^+\right] q^- = \left[2\omega\mathcal{N}, \phi_N^+\right] q^- = 2\omega\phi_N^+ q^-.$$

For the term containing $\hat{q}^+$ we need the superHamiltonian (14) with ω replaced by $-\omega$:

$$\hat{\mathcal{H}} = \mathcal{H} - 4\omega\mathcal{N} + 2\omega(1 + (N-1)(Nl+1)) \qquad [\hat{\mathcal{H}}, \hat{q}^{\pm}] = 0.$$

Then we can proceed with the commutators:

$$[\mathcal{H}, \hat{q}^+\phi_N] = [\hat{\mathcal{H}} + 4\omega\mathcal{N}, \hat{q}^+\phi_N] = [\hat{\mathcal{H}}, \hat{q}^+\phi_N] = \hat{q}^+[\hat{\mathcal{H}}, \phi_N] = \hat{q}^+[-2\omega\mathcal{N}, \phi_N] = 2\omega\hat{q}^+\phi_N.$$

Finally,

$$\left[\mathcal{H}, \mathcal{N}Q_N^+\right] = \left[\mathcal{H}, \phi_N^+ Q_C^-\right] = \left[\mathcal{H}, \phi_N^+\right] Q_C^- = 2\omega\phi_N^+ Q_C^- = 2\omega\mathcal{N}Q_N^+.$$

If we now recall (63), we see that all the terms in the operator (69) commute with the superHamiltonian in accordance with (28), so the latter is true.

Note that from (63) it follows that

$$[h, \mathcal{D}^{\pm}] = \pm 2\omega\mathcal{D}^{\pm} \tag{71}$$

where $h = \mathcal{H} - H_C$ is the CM-independent part of the Calogero superHamiltonian (41), where H_C is given by (40).

Same as (28), equation (71) describes an oscillatory algebra and hence can be used for the construction of the spectrum of the superHamiltonian h and proof of its integrability. Namely, from (29) one can derive the ground-state wavefunction for h:

$$\psi_0 = \exp\left[-\frac{\omega}{2}\left(\sum_{j=1}^{N} x_j^2 - y_N^2\right)\right]\prod_{i<k}^{N}|x_i - x_j|^l.$$

Applying powers of the operators (66) and (67) $q^{\pm}$ from (37), and $\hat{q}^{\pm}$ from (68) to this wavefunction one can get the excited states of h, which parallels the construction from subsection 2.3.

Equations (69) and (70) are also useful for the derivation of (31):

$$H^{(0)} = \mathrm{Ts}\mathbf{L}_1 = \mathrm{Ts}\mathbf{L}_2 + \text{const}.$$

The first equality of (31) can be proved in the following way: taking into account (46), we get

$$\begin{aligned}
\mathrm{Ts}\mathbf{L}_1 = \mathrm{Ts}(\mathbf{L}^+\mathbf{L}^-) &= N\langle N|\mathcal{L}^+\mathcal{L}^-|N\rangle = N\langle N|\left[\mathcal{D}^+ + \mathrm{i}N^{-1/2}\left(\phi_N^+ q^- - \hat{q}^+\phi_N - \mathcal{N}Q_N^+\right)\right] \\
&\quad\times\left[\mathcal{D}^- + \mathrm{i}N^{-1/2}\left(\phi_N^+\hat{q}^- - q^+\phi_N - \mathcal{N}Q_N^-\right)\right]|N\rangle \\
&= N\langle N|\mathrm{i}N^{-1/2}\left[\phi_N^+ q^- - Q_N^+\right]\mathrm{i}N^{-1/2}[-q^+\phi_N - Q_N^-]|N\rangle \\
&= -\langle N| - \phi_N^+ q^- q^+\phi_N + Q_N^+ Q_N^-|N\rangle = \langle 0|q^+q^- + Q_C^- Q_C^+|0\rangle \\
&= \langle 0|h + H_C|0\rangle = H^{(0)}.
\end{aligned}$$

The second equality of (31) can be proved in a similar way:

$$\begin{aligned}
\mathrm{Ts}\mathbf{L}_2 = \mathrm{Ts}(\mathbf{L}^-\mathbf{L}^+) &= N\langle N|\mathcal{L}^-\mathcal{L}^+|N\rangle = N\langle N|\left[\mathcal{D}^- + \mathrm{i}N^{-1/2}\left(\phi_N^+\hat{q}^- - q^+\phi_N - \mathcal{N}Q_N^-\right)\right] \\
&\quad\times\left[\mathcal{D}^+ + \mathrm{i}N^{-1/2}\left(\phi_N^+ q^- - \hat{q}^+\phi_N - \mathcal{N}Q_N^+\right)\right]|N\rangle \\
&= N\langle N|\mathrm{i}N^{-1/2}\left[\phi_N^+\hat{q}^- - Q_N^+\right]\mathrm{i}N^{-1/2}[-\hat{q}^+\phi_N - Q_N^-]|N\rangle \\
&= -\langle N| - \phi_N^+\hat{q}^-\hat{q}^+\phi_N + Q_N^- Q_N^+|N\rangle = \langle 0|\hat{q}^+\hat{q}^- + \hat{Q}_C^-\hat{Q}_C^+|0\rangle \\
&= \langle 0|\hat{h} + \hat{H}_C|0\rangle = \hat{H}^{(0)} = H^{(0)} + 2\omega[1 + (N-1)(Nl+1)]
\end{aligned}$$

where the hat indicates the inversion of the sign of ω. We have used the fact that $\mathcal{D}^{\pm}|N\rangle = 0$; $\langle N|\mathcal{D}^{\pm} = 0$, because $D^{(0)\pm} = 0$.

We see that (31) actually follows from the two supersymmetries of the model.

4.4. The extension onto the root systems other than A_N

In the papers [13–17] Dunkl operators for the root systems other than A_N were introduced. The formalizm of the present text can be extended to these more general models; in particular, one can define analogues of the formulae (47)–(51). For the partial case (54) of the Clebsh–Gordan coefficient, analogues of the operators (58) and (62) that commute with the superHamiltonian can be considered.

For the construction of the super Lax operators for general root systems, one should use the formalism of [28] (bearing in mind appendix A from the present text). Then it would be interesting to see the relation between the analogues of operators (19) and (58) in this more general case (i.e., generalization of (57) and (61)).

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Appendix A

In this appendix, we are to prove that for any $V_{km} = -V_{mk}$ it is true that

$$\sum_{m\neq k} V_{km}\psi_k^+\psi_k\mathcal{K}_{km} = \sum_{m\neq k} V_{km}\psi_k^+\psi_m. \tag{A.1}$$

Proof.

$$\begin{aligned}\sum_{m\neq k} V_{km}\psi_k^+\psi_k\mathcal{K}_{km} &= \sum_{m\neq k} V_{km}\psi_k^+\psi_k\big[\psi_k^+\psi_m + \psi_m^+\psi_k - \psi_k^+\psi_k - \psi_m^+\psi_m + 1\big] \\ &= \sum_{m\neq k} V_{km}\big[\big(1-\psi_k\psi_k^+\big)\psi_k^+\psi_m + \psi_k^+\big(\delta_{km} - \psi_m^+\psi_k\big)\psi_k + \big(1-\psi_k\psi_k^+\big)\psi_k^+\psi_k \\ &\quad - \psi_k^+\psi_k\psi_m^+\psi_m + \psi_k^+\psi_k\big] = \sum_{m\neq k} V_{km}\big[\psi_k^+\psi_m - \psi_k^+\psi_k\psi_m^+\psi_m\big] = \sum_{m\neq k} V_{km}\psi_k^+\psi_m\end{aligned}$$

because the contraction of a symmetric object $\psi_k^+\psi_k\psi_m^+\psi_m$ and antisymmetric V_{km} is zero. □

Appendix B

In this appendix, we shall prove the following statement: for any i, j

$$\big(\widetilde{T}_{ij}^L\big)_{\alpha\xi}\big(\widetilde{T}_{ij}^{(M)}\big)_{\gamma\beta}\langle\zeta|C_\xi|\beta\rangle = \langle\nu|C_\alpha|\gamma\rangle\big(\widetilde{T}_{ij}^{(M)}\big)_{\nu\zeta} \tag{B.1}$$

where $\mathbf{T}_{ij}^L$ is defined in (56); $\mathbf{T}_{ij}^{(M)}$ in (44); C_ξ in (54).

We will need an auxiliary statement:

$$\mathcal{K}_{ij}C_\beta = C_\alpha\big(T_{ij}^L\big)_{\alpha\beta}\mathcal{K}_{ij}. \tag{B.2}$$

Proof of (B.2): it follows from (12) and (13) that

$$\mathcal{K}_{ij}\psi_i^+\psi_i = \psi_j^+\psi_j\mathcal{K}_{ij} \qquad \mathcal{K}_{ij}\psi_k^+\psi_k = \psi_k^+\psi_k\mathcal{K}_{ij} \qquad k\neq i, j$$

(no summation over repeated indices). Hence, one can check that

$$\mathcal{K}_{ij}\psi_k^+\psi_k = \psi_l^+\psi_l\big(T_{ij}^{(1)}\big)_{lk}\mathcal{K}_{ij}$$

where the summation is only over l, and the matrix $\mathbf{T}_{ij}^{(1)}$ is defined in (17). Then it follows that

$$\mathcal{K}_{ij}C_\beta = \mathcal{K}_{ij}R_{\beta k}\psi_k^+\psi_k = R_{\beta k}\big(T_{ij}^{(1)}\big)_{lk}\psi_l^+\psi_l\mathcal{K}_{ij} = R_{\beta k}\big(T_{ij}^{(1)}\big)_{lk}R_{ml}R_{mn}\psi_n^+\psi_n\mathcal{K}_{ij}$$
$$= \big(T_{ij}^L\big)_{m\beta}C_m\mathcal{K}_{ij} = \big(T_{ij}^L\big)_{\mu\beta}C_\mu\mathcal{K}_{ij}$$

where we have used the identities:

$$R_{\beta k}\big(T_{ij}^{(1)}\big)_{lk}R_{ml} = \big(T_{ij}^L\big)_{m\beta} \qquad \big(T_{ij}^L\big)_{N\beta} = 0$$

proved in [38].

Now we can use (B.2) to prove (B.1):

$$\big(T_{ij}^L\big)_{\alpha\xi}\big(T_{ij}^{(M)}\big)_{\gamma\beta}\langle\zeta|C_\xi|\beta\rangle = \big(T_{ij}^L\big)_{\alpha\xi}\langle\zeta|C_\xi|\beta\rangle\langle\beta|\mathcal{K}_{ij}|\gamma\rangle = \langle\zeta|C_\xi\big(T_{ij}^L\big)_{\xi\alpha}\mathcal{K}_{ij}|\gamma\rangle$$
$$= \langle\zeta|\mathcal{K}_{ij}C_\alpha|\gamma\rangle = \langle\zeta|\mathcal{K}_{ij}|\nu\rangle\langle\nu|C_\alpha|\gamma\rangle = \langle\nu|C_\alpha|\gamma\rangle\big(T_{ij}^{(M)}\big)_{\nu\zeta}.$$

Appendix C

In this appendix, we are to prove that

$$D_{\sigma\alpha}^{(M)} = \langle\sigma|C_\xi|\beta\rangle R_{\xi k}\left[-\mathrm{i}\partial_k\delta_{\beta\alpha} + \mathrm{i}\sum_{m\neq k}V_{km}\big(\widetilde{T}_{km}^{(M)}\big)_{\beta\alpha}\right]$$
$$= \langle\sigma|\mathcal{L}|\alpha\rangle + \mathrm{i}N^{-1/2}M\frac{\partial}{\partial y_N}\delta_{\sigma\alpha} \tag{C.1}$$

where $\mathcal{L}$ is the super Lax operator (19).

Proof. One can modify the first line of (C.1) in the following way:

$$D_{\sigma\alpha}^{(M)} = \langle\sigma|C_\xi|\beta\rangle R_{\xi k}\left[-\mathrm{i}\partial_k\delta_{\beta\alpha} + \mathrm{i}\sum_{m\neq k}V_{km}\big(\widetilde{T}_{km}^{(M)}\big)_{\beta\alpha}\right]$$
$$= \langle\sigma|C_\xi|\beta\rangle R_{\xi k}\langle\beta| - \mathrm{i}\partial_k + \mathrm{i}\sum_{m\neq k}V_{km}\mathcal{K}_{km}|\alpha\rangle$$
$$= \langle\sigma|C_\xi R_{\xi k}|\beta\rangle\langle\beta| - \mathrm{i}\partial_k + \mathrm{i}\sum_{m\neq k}V_{km}\mathcal{K}_{km}|\alpha\rangle. \tag{C.2}$$

Taking into account definition (54) and the orthogonality of $\mathbf{R}$, one can see that

$$C_\xi R_{\xi k} = R_{\xi k}R_{\xi l}\psi_l^+\psi_l = (\delta_{kl} - R_{Nk}R_{Nl})\psi_l^+\psi_l = \psi_k^+\psi_k - N^{-1}\sum_l\psi_l^+\psi_l \tag{C.3}$$

where no summation over k is implied. Thus,

$$\langle\sigma|C_\xi R_{\xi k}|\beta\rangle = \langle\sigma|\psi_k^+\psi_k - N^{-1}\sum_l\psi_l^+\psi_l|\beta\rangle = \langle\sigma|\psi_k^+\psi_k - N^{-1}M|\beta\rangle. \tag{C.4}$$

Plugging (C.4) into (C.2), we get

$$D_{\sigma\alpha}^{(M)} = \langle\sigma|\psi_k^+\psi_k - N^{-1}M|\beta\rangle\langle\beta| - \mathrm{i}\partial_k + \mathrm{i}\sum_{m\neq k}V_{km}\mathcal{K}_{km}|\alpha\rangle = \langle\sigma|\big(\psi_k^+\psi_k - N^{-1}M\big)$$
$$\times\left[-\mathrm{i}\partial_k + \mathrm{i}\sum_{m\neq k}V_{km}\mathcal{K}_{km}\right]|\alpha\rangle = \langle\sigma|\left[-\mathrm{i}\psi_k^+\psi_k\partial_k + \mathrm{i}\sum_{m\neq k}V_{km}\psi_k^+\psi_k\mathcal{K}_{km}\right.$$
$$\left. + \mathrm{i}N^{-1}M\sum_k\partial_k - \mathrm{i}N^{-1}M\sum_{m\neq k}V_{km}\mathcal{K}_{km}\right]|\alpha\rangle. \tag{C.5}$$

Plugging (A.1) into (C.5) and taking into account that the contraction of a symmetric object $\mathcal{K}_{km}$ and antisymmetric V_{km} is always zero, we get

$$D_{\sigma\alpha}^{(M)} = \langle\sigma| \left[-\mathrm{i}\psi_k^+\psi_k\partial_k + \mathrm{i}\sum_{m\neq k} V_{km}\psi_k^+\psi_m + \mathrm{i}N^{-1/2}M\frac{\partial}{\partial y_N} \right] |\alpha\rangle$$
$$= \langle\sigma|\mathcal{L}|\alpha\rangle + \mathrm{i}N^{-1/2}M\frac{\partial}{\partial y_N}\delta_{\sigma\alpha}. \qquad \square$$

Appendix D

In this appendix, we will determine the form of the operator (58). Plugging (57) into (58), we get

$$\mathcal{D} = \sum_{M,\sigma,\alpha} \langle\sigma|\mathcal{L}|\alpha\rangle\big[|\sigma\rangle\langle\alpha| + |N\sigma\rangle\langle\alpha N|\big] + \mathrm{i}N^{-1/2}\frac{\partial}{\partial y_N}\sum_{M,\sigma} M\big[|\sigma\rangle\langle\sigma| + |N\sigma\rangle\langle\sigma N|\big]$$
$$\equiv \mathcal{D}^{(1)} + \mathcal{D}^{(2)}. \tag{D.1}$$

One could rewrite the operator $\mathcal{D}^{(2)}$ as

$$\mathcal{D}^{(2)} = \mathrm{i}N^{-1/2}\frac{\partial}{\partial y_N}\sum_{M,\sigma} M\big[|\sigma\rangle\langle\sigma| + |N\sigma\rangle\langle\sigma N|\big]$$
$$= \mathrm{i}N^{-1/2}\frac{\partial}{\partial y_N}\big(\mathcal{S} + \phi_N^+\mathcal{S}\phi_N\big) \tag{D.2}$$

where

$$\mathcal{S} \equiv \sum_{M,\sigma} M|\sigma\rangle\langle\sigma| = \sum_{M,\sigma,\beta}\phi_\beta^+\phi_\beta|\sigma\rangle\langle\sigma| = \sum_\beta \phi_\beta^+\phi_\beta\big(1 - \phi_N^+\phi_N\big). \tag{D.3}$$

It follows from (D.3) that

$$\phi_N^+\mathcal{S}\phi_N = \phi_N^+\sum_\beta\phi_\beta^+\phi_\beta\big(1-\phi_N^+\phi_N\big)\phi_N = \phi_N^+\sum_\beta\phi_\beta^+\phi_\beta\phi_N = \sum_\beta\phi_\beta^+\phi_\beta\phi_N^+\phi_N. \tag{D.4}$$

Plugging (D.3) and (D.4) into (D.2), one obtains

$$\mathcal{D}^{(2)} = \mathrm{i}N^{-1/2}\frac{\partial}{\partial y_N}\sum_\beta\phi_\beta^+\phi_\beta. \tag{D.5}$$

To get an explicit form of $\mathcal{D}^{(1)}$ in (D.1), note: for any operator $\mathcal{A}$ of the form (21),

$$\sum_{M,\sigma,\alpha}\langle\sigma|\mathcal{A}|\alpha\rangle\big[|\sigma\rangle\langle\alpha| + |N\sigma\rangle\langle\alpha N|\big] = \mathcal{A} - N^{-1/2}\sum_{k,m}A_{km}\big[\psi_k^+\phi_N + \phi_N^+\psi_m\big]$$
$$+ N^{-1}\phi_N^+\phi_N\sum_{k,m}A_{km} \tag{D.6}$$

where $\mathcal{N}|\sigma\rangle = M|\sigma\rangle$; $\mathcal{N}|\alpha\rangle = M|\alpha\rangle$. The proof of this relation is rather long, and we will not give it. In short, it uses the following auxillary relation:

$$\sum_{M,\sigma,\alpha}\langle\sigma|\mathcal{A}|\alpha\rangle[|\sigma\rangle\langle\alpha| + |N\sigma\rangle\langle\alpha N|] = \mathcal{A} + \big[\phi_N^+,\mathcal{A}\big]\phi_N - \phi_N^+[\phi_N,\mathcal{A}] + \big\{\phi_N^+,[\phi_N,\mathcal{A}]\big\}\phi_N^+\phi_N$$

that is true for *any* fermionic operator $\mathcal{A}$, not necessarily bilinear, and follows from the completeness of the basis (42) and anticommutation relations (36).

Taking into account (D.6) for the operator $\mathcal{L}$, we get

$$\mathcal{D}^{(1)} = \mathcal{L} - N^{-1/2}\sum_{k,m} L_{km}\big[\psi_k^+\phi_N + \phi_N^+\psi_m\big] + N^{-1}\phi_N^+\phi_N\sum_{k,m} L_{km}. \tag{D.7}$$

Since the super Lax operator $\mathcal{L}$ has the form (19), one can check that [27]

$$\sum_m L_{km} = -\mathrm{i}Q_k^- \qquad \sum_k L_{km} = \mathrm{i}Q_m^+ \qquad \sum_{k,m} L_{km} = -\mathrm{i}N^{1/2}\frac{\partial}{\partial y_N} \tag{D.8}$$

where the operators $Q_i^\pm$ are defined by (9) and y_N by (35). Therefore,

$$\sum_{km} L_{km}\psi_k^+ = -\mathrm{i}Q^+ \qquad \sum_{km} L_{km}\psi_m = \mathrm{i}Q^- \tag{D.9}$$

where $Q^\pm$ are supercharges (3) with bosonic part (14).

Plugging (D.8) and (D.9) into (D.7) we get

$$\mathcal{D}^{(1)} = \mathcal{L} + \mathrm{i}N^{-1/2}\left[Q^+\phi_N - \phi_N^+Q^- - \phi_N^+\phi_N\frac{\partial}{\partial y_N}\right]. \tag{D.10}$$

Bringing together (D.10), (D.5) and (D.1) one sees that

$$\begin{aligned}\mathcal{D} &= \mathcal{L} + \mathrm{i}N^{-1/2}\left[Q^+\phi_N - \phi_N^+Q^- + \left(\sum_\beta \phi_\beta^+\phi_\beta - \phi_N^+\phi_N\right)\frac{\partial}{\partial y_N}\right]\\ &= \mathcal{L} + \mathrm{i}N^{-1/2}\left[Q^+\phi_N - \phi_N^+Q^- + \big(\mathcal{N} - 2\phi_N^+\phi_N\big)\frac{\partial}{\partial y_N}\right].\end{aligned}$$

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